Homework 4 Solutions 2024-2025

The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Probability and Stochastic Analysis
Prepared by Tianxu Lan
Please send corrections, if any, to 1155184513@link.cuhk.edu.hk

February 18, 2025

1.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$.

(a)

Let τ_1, τ_2 be two $\mathcal{F}$-stopping times. Prove that

$$\tau_1 \wedge \tau_2 := \min(\tau_1, \tau_2) \quad \text{and} \quad \tau_1 \vee \tau_2 := \max(\tau_1, \tau_2)$$

are both stopping times.

Solution.

$$\{\tau_1 \wedge \tau_2 \leq n\} = \{\tau_1 \leq n \text{ or } \tau_2 \leq n\} = \{\tau_1 \leq n\} \cup \{\tau_2 \leq n\} \in \mathcal{F}_n$$
$$\{\tau_1 \vee \tau_2 \leq n\} = \{\tau_1 \leq n \text{ and } \tau_2 \leq n\} = \{\tau_1 \leq n\} \cap \{\tau_2 \leq n\} \in \mathcal{F}_n$$

(b)

Let τ be an $\mathcal{F}$-stopping time. Prove that $\tau + 1$ is also an $\mathcal{F}$-stopping time.

Solution.

$$\{\tau + 1 \leq n\} = \{\tau \leq n - 1\} \in \mathcal{F}_{n-1} \subseteq \mathcal{F}_n.$$

2.

Let $X_0 = 0$, $X_n = \sum_{k=1}^n \xi_k$, where $(\xi_k)_{k\geq 1}$ is a sequence of independent and identically distributed random variables such that $P[\xi_k = \pm 1] = \frac{1}{2}$. Let M and N be two positive integers and define

$$\tau := \min\{n \geq 0 : X_n = -N \text{ or } X_n = M\}.$$

(a)

Prove that τ is an $\mathcal{F}$-stopping time, where $\mathcal{F}$ is the natural filtration generated by X.

Solution.

The result follows by taking $B = \{-N, M\}$ in lemma 2.15 of the lecture notes. You can also prove this explicitly by imitating the proof of lemma 2.15.

(b)

Assume that $\tau < +\infty$ a.s., prove that $P[X_\tau \in \{-N, M\}] = 1$.

Solution.

This is clear from the definition of τ. To prove it more rigorously, observe that

$$1 = P(\tau < \infty) \leq P(X_\tau \in \{-N, M\}) \leq 1.$$

(c)

Under the condition of (b), compute $E[X_\tau]$ and $P[X_\tau = -N]$.

Hint: Let X be a martingale and τ be a stopping time with respect to a filtration $\mathcal{F}$, and if $\tau < \infty$ and the process $(X_{\tau\wedge n})_{n\geq 0}$ is uniformly bounded, then $E[X_\tau] = E[X_0]$.

Solution.

(a) We already know that X is a martingale due to example 2.8 in the lecture notes; following the hint, we shall show that $X_{\tau\wedge n}$ is uniformly bounded and hence that $E[X_\tau] = E[X_0] = 0$.

If $\tau \leq n$, then

$$|X_{\tau\wedge n}| = |X_\tau| \leq \max\{N, M\}.$$

If $\tau > n$, then one of the following occurs: - $X_n < -N$ - $X_n \in (-N, M)$ - $X_n > M$

The first and the third possibilities cannot happen because, for example, if $X_n > M$, then there must exist some earlier time $k < n$ such that $X_k = M$, which contradicts the definition of τ. Therefore, we have that

$$|X_{\tau\wedge n}| = |X_n| \leq \max\{N, M\}, \text{ if } \tau > n.$$

This shows that $X_{\tau\wedge n}$ is uniformly bounded.

To find $P(X_\tau = -N)$, note that

$$0 = E[X_\tau] = -N \cdot P(X_\tau = -N) + M \cdot P(X_\tau = M)$$

$$1 = P[X_\tau \in \{-N, M\}] = P(X_\tau = -N) + P(X_\tau = M).$$

This implies that

$$P(X_\tau = -N) = \frac{M}{M+N}.$$